

Comment on “Quantum Games and Quantum Strategies”

In a recent Letter, Eisert *et al.* [1] presented a quantum mechanical generalization of Prisoner’s Dilemma. In the classical form of this game, rational analysis leads the two players to ‘defect’ against one-another in a mutually destructive fashion [2]. A central result of Eisert *et al.*’s Letter is the observation that their quantum variant, illustrated in Figure 1, can avoid the ‘dilemma’: the mutually destructive outcome is replaced with an effectively cooperative one. Specifically, it is asserted that the maximally entangled game’s unique Nash equilibrium [2] occurs when both players apply the strategy $\hat{Q} = i\sigma_z$, yielding a pay-off equivalent to cooperative behaviour in the classical game.

In this Comment we show that their observation is incorrect. The mistake follows from the following erroneous assertion:

It proves to be sufficient to restrict the strategic space to the 2-parameter set of unitary 2×2 matrices,

$$\hat{U}(\theta, \phi) = \begin{pmatrix} e^{i\phi}\cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & e^{-i\phi}\cos\theta/2 \end{pmatrix}, \quad (1)$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi/2$.

Here we explicitly consider the complete set of *all* local unitary operations (*i.e.* all of $SU(2)$), finding that the properties of the game are wholly different: the strategy \hat{Q} is not an equilibrium; indeed, there is *no equilibrium* in the space of deterministic quantum strategies. Moreover, it seems unlikely that the restriction to the set $\hat{U}(\theta, \phi)$ can reflect any reasonable physical constraint (limited experimental resources, say) because this set is not closed under composition. An ideal counter strategy to \hat{Q} , for example, is $i\sigma_x$, which is equal to $\hat{U}(0, \pi/2)\hat{U}(\pi, 0)$. The game of [1] therefore does not constitute a reasonable variant of the general case we consider here.

We will write the operations applied by the players in the form $\hat{A} \otimes \hat{B}$, where \hat{A} is applied to the qubit controlled by *A* and similarly for \hat{B} . Suppose that

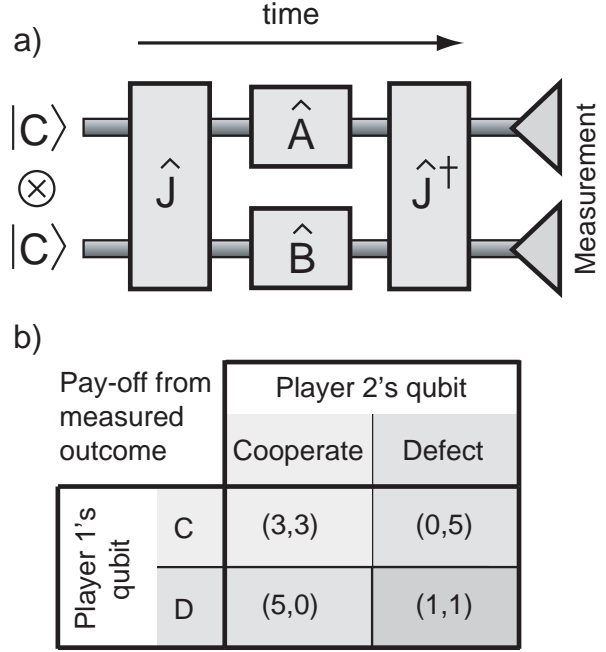


Figure 1: a) The quantized Prisoner’s Dilemma, as described in [1]. The pair of qubits are prepared in the unentangled state $|CC\rangle$ and then sent through the entangling gate \hat{J} . Players *A* and *B* then apply their local unitary operations $\hat{A} \otimes \hat{I}$ and $\hat{I} \otimes \hat{B}$, respectively. A gate inverse to \hat{J} is applied before the final measurement. b) The Prisoner’s Dilemma pay-off table chosen in [1].

player *A* applies transformation \hat{X} to her qubit, prepared as the first qubit in the maximally entangled state

$$\hat{J}|CC\rangle = \frac{1}{\sqrt{2}}(|CC\rangle + i|DD\rangle), \quad (2)$$

where $\hat{J} = \exp\{i\pi\hat{D} \otimes \hat{D}/4\}$ and $\hat{D} = i\sigma_y$ is the ‘defect’ matrix of [1]. The most general $\hat{X} \in SU(2)$ is of the form $\hat{X} = (x_{ij})$, where $x_{11} = x_{22}^*$, $x_{12} = -x_{21}^*$ and $\det \hat{X} = 1$. Therefore, *A* produces the state $(\hat{X} \otimes \hat{I})\hat{J}|CC\rangle = (I \otimes \hat{Y})\hat{J}|CC\rangle$ for $\hat{Y} = (y_{ij}) \in SU(2)$, where $y_{11} = x_{11}$ and $y_{12} = ix_{12}$. In other words, any unitary transformation which *A* applies locally to her qubit is actually equivalent to a unitary

transformation applied locally by B . Consequently, if B were to choose $\hat{D}\hat{Y}^\dagger$, we would have a final state $\hat{J}^\dagger(\hat{X} \otimes \hat{D}\hat{Y}^\dagger)\hat{J}|CC\rangle = \hat{J}^\dagger(\hat{I} \otimes \hat{D}\hat{Y}^\dagger\hat{Y})\hat{J}|CC\rangle = |CD\rangle$, the optimal outcome for B . Thus, for any given strategy of A , there is an ideal counter-strategy for B , and vice-versa; there is no Nash equilibrium of the kind suggested by Eisert *et al.* [3].

To obtain such equilibria we must extend the abilities of the players: it suffices to allow them to make probabilistic choices (rather than the full formalism of completely positive maps considered in footnote 14 of [1]). Suppose that A adopts the following strategy: she will choose a move $\hat{X} \in SU(2)$ at random with respect to the Haar measure on $SU(2)$. If B responds with $\hat{Y}_0 \in SU(2)$ then the probability that outcome $i \in \{CC, CD, DC, DD\}$ will be measured is

$$\begin{aligned} P_i(\hat{Y}_0) &= \int_{SU(2)} \left| \langle i | \hat{J}^\dagger(\hat{X} \otimes \hat{Y}_0)\hat{J} | CC \rangle \right|^2 d\hat{X} \\ &= \int_{SU(2)} \left| \langle i | \hat{J}^\dagger(\hat{X} \hat{X}_0^\dagger \hat{X}_0 \otimes I)\hat{J} | CC \rangle \right|^2 d\hat{X} \\ &= \int_{SU(2)} \left| \langle i | \hat{J}^\dagger(\hat{X} \otimes \hat{I})\hat{J} | CC \rangle \right|^2 d\hat{X} \\ &= P_i(\hat{I}), \end{aligned} \quad (3)$$

where $\hat{X}_0 \in SU(2)$ is chosen such that $(\hat{X}_0 \otimes \hat{I})\hat{J}|CC\rangle = (\hat{I} \otimes \hat{Y}_0)\hat{J}|CC\rangle$ and we have used the right invariance of the Haar measure, assumed to be normalized such that the volume of $SU(2)$ is 1. Thus, B 's choice of strategy does not matter; regardless of his choice, his expected pay-off is simply an unbiased average over the classical pay-offs. Therefore, the situation where both players adopt this random strategy is a Nash equilibrium: neither player can improve his or her payoff by unilaterally altering choice of strategy.

As a final point, we note that one can construct Prisoner's Dilemma-type pay-off tables for which the quantum equilibrium pay-off we describe above is below the classical equilibrium pay-off, or above it, or even above the classical cooperative pay-off. In this last case the 'dilemma' may be said to have been removed [4]. To this extent the behavior of the quantum generalization is qualitatively different from the classical case, although in a way that is perhaps less

surprising than originally suggested by Eisert *et al.*

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References

- [1] J. Eisert, M. Wilkens, M. Lewenstein, *Phys. Rev. Lett.* **83** 3077 (1999), quant-ph/9806088.
- [2] R. B. Myerson, *Game Theory: An Analysis of Conflict* (MIT Press, Cambridge, MA, 1991).
- [3] This result is a familiar property of maximally entangled states and is easily generalized to two maximally entangled n -dimensional systems \mathcal{H}_A and \mathcal{H}_B . Such states can always be written in the form

$$|\Psi\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^n |i\rangle_A \otimes |i\rangle_B, \quad (4)$$

where ${}_A \langle i | j \rangle_A = {}_B \langle i | j \rangle_B = \delta_{ij}$. It is then easy to check that if $U \in SU(\mathcal{H}_A)$, $U \otimes I |\Psi\rangle = I \otimes U^T |\Psi\rangle$. Since the special unitary group is closed under transpose, it follows that our observations on the present quantum game, the classical form of which involves the one-bit strategies $\{C, D\}$, will also apply to quantum generalizations of any larger classical two-player symmetric games.

- [4] For the pay-off values (3,5,0,1) given in [1], the quantum equilibrium pay-off (QP) is 2.25, in-between the classical equilibrium pay-off (CEP) of 1 and the classical cooperative pay-off (CCP) of 3. However, for the values (5,6,0,4) the QEP is below the CEP, whilst for (3,9,0,1) it is above the CCP.